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SYSTEMS OF LINEAR INEQUALITIES.

By Lloyd L. Dines.

The existence of a solution of the system of inequalities

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n > 0\\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n > 0\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
&a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n > 0,
\end{aligned}
\tag{1}
$$

and the character of the solution in case one exists, obviously depend upon the matrix of the coefficients. There seems however to be no terminology available for the characterization of the solution in terms of this matrix, and no well-recognized algorithm for the actual determination of the solution. This is in marked contrast to the analogous situation in the case of the system of equations formed by replacing the symbol $>$ by the symbol $=$ in (1). In the present paper a study is made of the system (1) and also of a system of non-homogeneous inequalities from the point of view of the matrix of coefficients. The central feature is a concept analogous to the rank of a matrix, which we call the *inequality-rank* or *I-rank* of the matrix. In terms of this concept the principal results are expressed in Theorems I–IV.

1. The I-rank of a Matrix. Let

$$
M = \left\| \begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{matrix} \right\|
$$

be a matrix whose mn elements are real. For our purpose it is important to distinguish the case in which at least one column of M has elements all of the same sign. We make the

Definition. *A matrix will be said to be I-positive* (*or I-negative*) *with respect to a given one of its columns if all elements of that column are positive* (*or negative*). *In either case the matrix will be said to be I-definite with respect to that column. A matrix will be said to be I-positive* (*or I-negative, or I-definite*) *if it possesses a column with respect to which it is I-positive* (*or I-negative, or I-definite*).

Suppose the matrix M is not I-definite with respect to the rth column. Then the elements of that column can be divided into three classes, viz.,

those which are positive: a_{ir}, $i = i_1, i_2, \cdots, i_P$;

those which are negative: a_{jr}, $j = j_1, j_2, \cdots, j_N$;

those which are zero: a_{kr}, $k = k_1, k_2, \cdots, k_Z$;

the number of elements in the respective classes being represented by P, N, and Z.

Consider a matrix $M_1^{(r)}$ derived from M as follows:

To each pair of elements a_{ir}, a_{jr}, the first positive and the second negative, corresponds one row of the derived matrix, the elements of which are second order determinants

$$\begin{vmatrix} a_{ir} & a_{i1} \\ a_{jr} & a_{j1} \end{vmatrix}, \quad \begin{vmatrix} a_{ir} & a_{i2} \\ a_{jr} & a_{j2} \end{vmatrix}, \quad \cdots, \quad \begin{vmatrix} a_{ir} & a_{ir-1} \\ a_{jr} & a_{jr-1} \end{vmatrix}, \quad \begin{vmatrix} a_{ir} & a_{ir+1} \\ a_{jr} & a_{jr+1} \end{vmatrix}, \quad \cdots, \quad \begin{vmatrix} a_{ir} & a_{in} \\ a_{jr} & a_{jn} \end{vmatrix}.$$

To each zero element a_{kr} corresponds one row of the derived matrix

$$a_{k1}, \quad a_{k2}, \quad \cdots, \quad a_{kr-1}, \quad a_{kr+1}, \quad \cdots, \quad a_{kn}.$$

The matrix $M_1^{(r)}$ will then consist of the rows so formed, their number being $P \cdot N + Z$. The sequential order of the rows will be determined by the convention that: (1) each (ij) row precedes every (k) row; and (2) of two (ij) rows that one precedes which has the smaller i or (in case of equal i's) that one which has the smaller j.

The matrix $M_1^{(r)}$ is then well-defined provided the matrix M is not I-definite with respect to the rth column. For the sake of uniformity it will be convenient to define $M_1^{(r)}$ in case M is I-definite with respect to the rth column, as the matrix of one row and $n - 1$ columns, all elements of which are $+1$ or -1 according as M is I-positive or I-negative with respect to the rth column. The matrix $M_1^{(r)}$ will be called the I-*complement* of the rth column of M, and may be represented by

$$M_1^{(r)} = \begin{Vmatrix} a_{11}^{(r)} & a_{12}^{(r)} & \cdots & a_{1r-1}^{(r)} & a_{1r+1}^{(r)} & \cdots & a_{1n}^{(r)} \\ a_{21}^{(r)} & a_{22}^{(r)} & \cdots & a_{2r-1}^{(r)} & a_{2r+1}^{(r)} & \cdots & a_{2n}^{(r)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m_r1}^{(r)} & a_{m_r2}^{(r)} & \cdots & a_{m_rr-1}^{(r)} & a_{m_rr+1}^{(r)} & \cdots & a_{m_rn}^{(r)} \end{Vmatrix}.$$

The n matrices $M_1^{(1)}, M_1^{(2)}, \cdots, M_1^{(n)}$ will be called the *I-minors* of $n - 1$ columns of the matrix M.

We may now form in a similar way the I-complement of any column of the matrix $M_1^{(r)}$. The I-complement of the column headed by $a_{1s}^{(r)}$ will be denoted by $M_2^{(rs)}$. The $n(n-1)$ matrices $M_2^{(rs)}$, all possible choices of r and s being considered, will be called the I-minors of $n-2$ columns of the matrix M. More generally, we define recursively $M_{p+1}^{(r_1r_2\cdots r_{p+1})}$ as the I-complement of that column of $M_p^{(r_1r_2\cdots r_p)}$ which is headed by the element $a_{1r_{p+1}}^{(r_1r_2\cdots r_p)}$. We thus have relative to the matrix M a well-defined system of I-minors $M_p^{(r_1r_2\cdots r_p)}$, $(p = 1, 2, \cdots, n-1)$. The matrix M itself may be considered as the I-minor of n columns.* A noteworthy property of the system thus defined is that if $M_p^{(r_1r_2\cdots r_p)}$ is I-definite, then $M_{p+1}^{(r_1r_2\cdots r_{p+1})}$ is I-definite.

DEFINITION: *A matrix will be said to be of I-rank k if it possesses at least one I-minor of k columns which is I-definite, but does not possess any I-minor of $k+1$ columns which is I-definite. If none of its I-minors are I-definite it will be said to be of I-rank zero.*

As an immediate consequence of this definition there follows the

COROLLARY: *The I-rank of a matrix is not altered if*

(1) *any two rows or any two columns are interchanged;*

(2) *all elements of any row or any column are multiplied by the same positive constant.*

2. Systems of Linear Homogeneous Inequalities. Relative to the system of inequalities

$$
\begin{array}{c}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n > 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n > 0 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n > 0,
\end{array}
\tag{1}
$$

the matrix of whose coefficients is M, we have the following theorems.

THEOREM I: *A necessary and sufficient condition for the existence of a solution of the system* (1) *is that the I-rank of the matrix M be greater than zero.*

THEOREM II: *If the I-rank of the matrix M is $k(>0)$, then the system* (1) *possesses a solution in which $k-1$ of the unknowns may be assigned values at pleasure.*

To prove Theorem II, suppose first that $k = n$. Then there is at least one column of M in which all elements have the same sign. We may suppose without loss of generality that the first column has this property, since in any case that condition could be brought about by a suitable ordering of the unknowns. The coefficients of x_1 in (1) are then all positive or all negative. In the former case the system is equivalent to

---

* The total number of I-minors of M is easily seen to be $1 + \sum_{k=1}^{n-1} \frac{n!}{(n-p)!}$.

$$
(2)\qquad
\begin{aligned}
x_1 &> -\frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3 - \cdots - \frac{a_{1n}}{a_{11}}x_n \\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
x_1 &> -\frac{a_{m2}}{a_{m1}}x_2 - \frac{a_{m3}}{a_{m1}}x_3 - \cdots - \frac{a_{mn}}{a_{m1}}x_n;
\end{aligned}
$$

and in the latter case to

$$
(3)\qquad
\begin{aligned}
x_1 &< -\frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3 - \cdots - \frac{a_{1n}}{a_{11}}x_n \\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
x_1 &< -\frac{a_{m2}}{a_{m1}}x_2 - \frac{a_{m3}}{a_{m1}}x_3 - \cdots - \frac{a_{mn}}{a_{m1}}x_n.
\end{aligned}
$$

In either case $x_2, x_3, \cdots, x_n$ can be assigned values at pleasure. The variable x_1 is merely required in the former case to exceed the m lower bounds defined by (2) and in the latter case to be less than the m upper bounds defined by (3).

Next suppose $k < n$. By hypothesis there exists a k-columned I-minor of M which is I-definite. We may without loss of generality suppose that it is $M_{n-k}{}^{(12\cdots n-k)}$. We divide the inequalities of (1) into three classes according as the coefficients of x_1 are positive, negative, or zero, and as in §1, denote the leading coefficients of the three classes by a_{i1}, a_{j1}, a_{k1} respectively. Then the system (1) is equivalent to the system

$$
\begin{aligned}
&(4_P)\quad x_1 > -\frac{a_{i2}}{a_{i1}}x_2 - \frac{a_{i3}}{a_{i1}}x_3 - \cdots - \frac{a_{in}}{a_{i1}}x_n, \qquad i = i_1, i_2, \cdots, i_P; \\
&(4_N)\quad x_1 < -\frac{a_{j2}}{a_{j1}}x_2 - \frac{a_{j3}}{a_{j1}}x_3 - \cdots - \frac{a_{jn}}{a_{j1}}x_n, \qquad j = j_1, j_2, \cdots, j_N; \\
&(4_Z)\quad 0 > -a_{k2}x_2 - a_{k3}x_3 - \cdots - a_{kn}x_n, \qquad k = k_1, k_2, \cdots, k_Z.
\end{aligned}
$$

By this system the variable x_1 is restricted to lie between the P lower bounds prescribed in (4_P) and the N upper bounds prescribed in (4_N). Possible values for x_1 will therefore exist if and only if the right side of each inequality of (4_P) is exceeded by the right side of every inequality of (4_N), that is if

$$
(5)\quad \left(\frac{a_{i2}}{a_{i1}} - \frac{a_{j2}}{a_{j1}}\right)x_2 + \left(\frac{a_{i3}}{a_{i1}} - \frac{a_{j3}}{a_{j1}}\right)x_3 + \cdots + \left(\frac{a_{in}}{a_{i1}} - \frac{a_{jn}}{a_{j1}}\right)x_n > 0, \qquad
\begin{aligned} i &= i_1, i_2, \cdots, i_P, \\ j &= j_1, j_2, \cdots, j_N. \end{aligned}
$$

We may then think of x_1 as having been *eliminated*, and proceed to a consideration of the conditions upon $x_2, x_3, \cdots, x_n$, imposed by (5) and (4_z).

Remembering that $-a_{i1} \cdot a_{j1} > 0$, we may write (5) in the equivalent form

$$(a_{i1}a_{j2} - a_{j1}a_{i2})x_2 + (a_{i1}a_{j3} - a_{j1}a_{i3})x_3 + \cdots + (a_{i1}a_{jn} - a_{j1}a_{in})x_n > 0.$$

These conditions, together with the conditions

$$a_{k2}x_2 + a_{k3}x_3 + \cdots + a_{kn}x_n > 0$$

imposed by (4_z), form a system

$$\begin{array}{ll}
 & a_{i2}^{(1)}x_2 + a_{i3}^{(1)}x_3 + \cdots + a_{in}^{(1)}x_n > 0 \\
(6) & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
 & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
 & a_{m_12}^{(1)}x_2 + a_{m_13}^{(1)}x_3 + \cdots + a_{m_1n}^{(1)}x_n > 0,
\end{array}$$

whose matrix is what we have called the I-minor $M_1^{(1)}$.

If $M_1^{(1)}$ is not I-definite (that is if $k < n - 1$), we proceed to *eliminate* x_2 from the system (6) as x_1 was eliminated from (1), obtaining a system of inequalities in the variables $x_3, x_3, \cdots, x_n$, whose matrix is $M_2^{(12)}$.

By successive application of the process indicated, the variables $x_1, x_2, \cdots, x_{n-k}$, may be eliminated. The result will be the system

$$\begin{array}{ll}
 & a_{1n-k+1}^{(v)}x_{n-k+1} + a_{1n-k+2}^{(v)}x_{n-k+2} + \cdots + a_{1n}^{(v)}x_n > 0 \\
(7) & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
 & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
 & a_{m_vn-k+1}^{(v)}x_{n-k+1} + a_{m_vn-k+2}^{(v)}x_{n-k+2} + \cdots + a_{m_vn}^{(v)}x_n > 0
\end{array}$$

(v) represents the sequence $(12 \cdots n-k)$,

whose matrix is $M_{n-k}^{(12 \cdots n-k)}$ and is therefore I-definite. We may suppose that the column with respect to which this matrix is I-definite is the first. Hence the system (7) is equivalent to one of the two systems

$$\begin{array}{ll}
 & x_{n-k+1} > -\dfrac{a_{1n-k+2}^{(v)}}{a_{1n-k+1}^{(v)}}x_{n-k+2} - \cdots - \dfrac{a_{1n}^{(v)}}{a_{1n-k+1}^{(v)}}x_n \\
(8) & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
 & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
 & x_{n-k+1} > -\dfrac{a_{m_vn-k+2}^{(v)}}{a_{m_vn-k+1}^{(v)}}x_{n-k+2} - \cdots - \dfrac{a_{m_vn}^{(v)}}{a_{m_vn-k+1}^{(v)}}x_n,
\end{array}$$

or

$$\begin{array}{ll}
 & x_{n-k+1} < -\dfrac{a_{1n-k+2}^{(v)}}{a_{1n-k+1}^{(v)}}x_{n-k+2} - \cdots - \dfrac{a_{1n}^{(v)}}{a_{1n-k+1}^{(v)}}x_n \\
(9) & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
 & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
 & x_{n-k+1} < -\dfrac{a_{m_vn-k+2}^{(v)}}{a_{m_vn-k+1}^{(v)}}x_{n-k+2} - \cdots - \dfrac{a_{m_vn}^{(v)}}{a_{m_vn-k+1}^{(v)}}x_n.
\end{array}$$

In either case, $x_{n-k+2}, x_{n-k+3}, \cdots, x_n$, can be assigned values at pleasure, and x_{n-k+1} will be limited by the lower bounds of (8) or the upper bounds of (9). The remaining variables $x_{n-k}, x_{n-k-1}, \cdots, x_1$ will have restricted but actually existent ranges defined by the systems of inequalities occurring in the elimination process, the system (4), defining the range for x_1 being typical. This completes the proof of Theorem II.

The *sufficiency* of the condition stated in Theorem I is implied by Theorem II. To prove the *necessity* of the condition, we note that the systems of inequalities presenting themselves in the successive steps of the process of elimination above described must necessarily be satisfied if a solution of (1) is to exist. If the I-rank of the matrix M is not greater than zero,.the process of elimination can be continued until a system

$$\begin{array}{c} a_{1n}{}^{(\mu)}x_n > 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ a^{(\mu)}_{m_\mu n} > 0 \end{array}$$

containing only one variable is obtained. Since the matrix of this system, possessing as it does only one column, is by hypothesis not I-definite, the system is obviously inconsistent. This completes the proof of Theorem I.

3. Systems of Linear Non-homogeneous Inequalities. Consider the system of non-homogeneous inequalities

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_1 > 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_2 > 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + b_m > 0, \end{array} \tag{10}$$

the matrix of whose coefficients is M.

By a process of elimination similar to that used in the preceding section, we may obtain

THEOREM III: *If the I-rank of the matrix M is $k(>0)$, then the system* (10) *possesses a solution in which $k - 1$ of the unknowns may be assigned values at pleasure.*

From this theorem it follows that a sufficient condition for the existence of a solution of (10) is that the I-rank of M be greater than zero. That this condition is not necessary may be shown by a simple example. The system

$$\begin{array}{c} x_1 + 1 > 0 \\ -x_1 + 1 > 0 \end{array}$$

has a matrix of I-rank zero, but has the obvious solution

$$-1 < x_1 < 1.$$

A condition which is both necessary and sufficient for the existence of a solution of (10) may be obtained as follows. In (10) we make the substitution

$$x_1 = \frac{x_1'}{x_{n+1}'}, \qquad x_2 = \frac{x_2'}{x_{n+1}'}, \qquad \cdots, \qquad x_n = \frac{x_n'}{x_{n+1}'}$$

with the restriction

$$x_{n+1}' > 0. \tag{11}$$

Upon multiplication of each inequality of (10) by x_{n+1}', and annexation of (11), there results the system of homogeneous inequalities:

$$\begin{aligned} a_{11}x_1' + a_{12}x_2' + \cdots + a_{1n}x_n' + b_1x_{n+1}' &> 0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot &\ \cdot \\ a_{m1}x_1' + a_{m2}x_2' + \cdots + a_{mn}x_n' + b_mx_{n+1}' &> 0 \\ x_{n+1}' &> 0. \end{aligned} \tag{12}$$

Every solution $(x_1', x_2', \cdots, x_n', x_{n+1}')$ of the system (12) affords a solution

$$(x_1, x_2, \cdots, x_n) = \left(\frac{x_1'}{x_{n+1}'}, \frac{x_2'}{x_{n+1}'}, \cdots, \frac{x_n'}{x_{n+1}'}\right)$$

of the system (10). And conversely, to every solution $(x_1, x_2, \cdots, x_n)$ of (10) there corresponds a solution

$$(x_1', x_2', \cdots, x_n', x_{n+1}') = (x_1, x_2, \cdots, x_n, 1)$$

of (12). But a necessary and sufficient condition for the existence of a solution of (12) is by Theorem I that the I-rank of the matrix

$$N = \begin{Vmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \\ 0 & 0 & \cdots & 0 & 1 \end{Vmatrix}$$

be greater than zero. Hence

THEOREM IV: *A necessary and sufficient condition for the existence of a solution of the system* (10) *is that the I-rank of the matrix N be greater than zero.*

4. The Determination of the Solution. In the process of elimination carried out in §2, it was assumed that since the I-rank of the matrix M was k, the I-minor $M_{n-k}{}^{(12\cdots n-k)}$ was I-definite. The effect of this assumption was to assure us that upon the elimination of the variables $x_1, x_2, \cdots, x_{n-k}$, in their natural order, the matrix of the resulting system would be I-definite, and $k - 1$ of the unknowns could be assigned values at pleasure. While this assumption was justifiable in the existence proof of §2, it must be recognized that for a given system of inequalities with numerical coefficients, the number of unknowns whose values are arbitrary in the solution depends upon the order in which the elimination is made.

Consider for example the simple system

$$x + y > 0$$
$$-x + y > 0.$$

If we solve first for x we obtain

$$-y < x < y$$

which implies the restriction $y > 0$.

But if we solve first for y, we obtain the solution

$$y > \begin{cases} x \\ -x \end{cases}, \qquad x \text{ arbitrary}$$

In the former case both variables are restricted; in the latter, one is arbitrary. Both solutions represent the totality of number pairs (x, y) which satisfy the given system of inequalities.*

If merely a determination of the totality of number sets $(x_1, x_2, \cdots, x_n)$ which satisfy a system of inequalities of form (1) is desired, it may be obtained by the process of elimination described in §2, the order of elimination being immaterial. The calculation required is merely the formation of a sequence of I-minors of the matrix of coefficients, each I-minor being the I-complement of a column of its predecessor, the process to be continued until an I-definite matrix is obtained, or until the sequence is automatically terminated by an I-minor of one column. The successive I-minors are the matrices of the successive systems of inequalities occurring in the elimination. If no one of them is I-definite, the given system admits no solution. If the last one is I-definite, the possible ranges for the variables can be determined from them by inspection.

Example: Solve the system

$$\begin{aligned} x_1 - 3x_2 + x_3 - x_4 &> 0 \\ 2x_1 - 3x_2 + x_3 + 2x_4 &> 0 \\ 3x_1 - 2x_2 + 3x_3 + x_4 &> 0 \\ -2x_1 + 2x_2 - x_3 + 2x_4 &> 0. \end{aligned}$$

---

* The difference in form of the solutions is immediately explained if the number pair (x, y) be interpreted as the coördinates of a variable point in a plane.

We form the sequence of I-minors

$$M = \left\|\begin{array}{rrrr} 1 & -3 & 1 & -1 \\ 2 & -3 & 1 & 2 \\ 3 & -2 & 3 & 1 \\ -2 & 2 & -1 & 2 \end{array}\right\|, \quad M_1{}^{(1)} = \left\|\begin{array}{rrr} -4 & 1 & 0 \\ -2 & 0 & 8 \\ 2 & 3 & 8 \end{array}\right\|, \quad M_2{}^{(12)} = \left\|\begin{array}{rr} 14 & 32 \\ 6 & 32 \end{array}\right\|.$$

Since $M_2{}^{(12)}$ is I-definite with respect to both columns either x_3 or x_4 may be arbitrary, the other being restricted in terms of it. From $M_1{}^{(1)}$, x_2 may be bounded in terms of x_3 and x_4; and from M, x_1 may be bounded in terms of x_2, x_3 and x_4. A complete solution is

$$x_4 \text{ arbitrary}, \qquad x_3 > \left\{\begin{array}{l} -\tfrac{16}{7}x_4 \\ -\tfrac{16}{3}x_4, \end{array}\right.$$

$$\left.-\tfrac{3}{2}x_3 - 4x_4\right\} < x_2 < \left\{\begin{array}{l} \tfrac{1}{4}x_3 \\ 4x_4 \end{array}\right.$$

$$\left.\begin{array}{r} 3x_2 - x_3 + x_4 \\ \tfrac{3}{2}x_2 - \tfrac{1}{2}x_3 - x_4 \\ -\tfrac{2}{3}x_2 - x_3 - x_4 \end{array}\right\} < x_1 < \left\{ x_2 - \tfrac{1}{2}x_3 + x_4.\right.$$

A solution in which *two* of the variables are arbitrary may be obtained by forming the I-minors

$$M = \left\|\begin{array}{rrrr} 1 & -3 & 1 & -1 \\ 2 & -3 & 1 & 2 \\ 3 & -2 & 3 & 1 \\ -2 & 2 & -1 & 2 \end{array}\right\|, \quad M_1{}^{(3)} = \left\|\begin{array}{rrr} -1 & -1 & 1 \\ 0 & -1 & 4 \\ -3 & 4 & 7 \end{array}\right\|.$$

$$x_1 \text{ and } x_2 \text{ arbitrary}, \qquad x_4 > \left\{\begin{array}{r} x_1 + x_2 \\ \tfrac{1}{4}x_2, \\ \tfrac{3}{7}x_1 - \tfrac{4}{7}x_2 \end{array}\right.$$

$$\left.\begin{array}{r} -x_1 + 3x_2 + x_4 \\ -2x_1 + 3x_2 - 2x_4 \\ -x_1 + \tfrac{2}{3}x_2 - \tfrac{1}{3}x_4 \end{array}\right\} < x_3 < \left\{ -2x_1 + 2x_2 + 2x_4.\right.$$

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